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Wild ramification and a vanishing cycles formula

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Abstract

In [Duke Math. J. 55 (1987) 629–659] K. Kato proved, using techniques from K-theory, a formula which compares the dimensions of the spaces of vanishing cycles in a finite morphism between formal germs of curves over a complete discrete valuation ring. To the best of my knowledge Kato's formula is explicit only in the case where this morphism is generically separable on the level of special fibres. In this note we prove, using formal patching techniques à la Harbater, an analogous explicit formula in the case of a Galois cover of degree p between formal germs of curves over a complete discrete valuation ring of unequal characteristic $(0, p)$ which *includes* the case where we have inseparability on the level of special fibres. The results of this paper play a key role in [math.AG/0106249] where is studied the semi-stable reduction of Galois covers of degree p of curves over a complete discrete valuation ring of unequal characteristics $(0, p)$, as well as the Galois action on these covers.

0. Introduction

Let R be a complete discrete valuation ring of unequal characteristic, with uniformiser π, fractions field K, and algebraically closed residue field k of characteristic p. In this note we investigate Galois covers of degree p between formal germs of R-curves at closed points. Our main result is a formula comparing the dimensions of the spaces of vanishing cycles at the corresponding closed points. More precisely, we have the following:

Theorem (3.4). *Let $\mathcal{X} := \operatorname{Spf} \widehat{\mathcal{O}}_x$ be the formal germ of an R-curve at a closed point x, with $\mathcal{X}_k$ reduced. Let $f : \mathcal{Y} \to \mathcal{X}$ be a Galois cover of degree p with $\mathcal{Y}$* normal *and*

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local. Assume that the special fibre $\mathcal{Y}_k := \mathcal{Y} \times_R k$ of $\mathcal{Y}$ is reduced. Let $\{\wp_i\}_{i\in I}$ be the minimal prime ideals of $\widehat{\mathcal{O}}_x$ which contain π, and let $\mathcal{X}_i := \operatorname{Spf} \widehat{\mathcal{O}}_{\wp_i}$ be the formal completion of the localisation of $\mathcal{X}$ at $\wp_i$. For each $i \in I$ the above cover f induces a torsor $f_i : \mathcal{Y}_i \to \mathcal{X}_i$ under a finite and flat R-group scheme G_i of rank p above the boundary $\mathcal{X}_i$. Let $(G_{k,i}, m_i, h_i)$ be the reduction type of f_i as defined in Definition 2.4*, here $G_{k,i} := G_i \times_R k$ is the special fibre of the group scheme G_i. Let y be the closed point of $\mathcal{Y}$. Then one has the following **local Riemann–Hurwitz formula**:*

$$2g_y - 2 = p(2g_x - 2) + d_\eta - d_s,$$

where g_y (respectively g_x) denotes the genus of the singularity at y (respectively x), d_η is the degree of the divisor of ramification in the morphism $f_\eta : \mathcal{Y}_\eta \to \mathcal{X}_\eta$ induced by f on the generic fibre, and $d_s := \sum_{i\in I^{\mathrm{rad}}}(m_i - 1)(p - 1) + \sum_{i\in I^{\mathrm{et}}}(m_i - 1)(p - 1)$, where I^{rad} is the subset of I consisting of those i for which $G_{k,i}$ is radical, and I^{et} is the subset of I consisting of those i for which $G_{k,i}$ is étale and $m_i \neq 0$.

In particular, the genus g_y of y depends only on the genus g_x of x, the ramification data on the generic fibre in the above morphism $f : \mathcal{Y} \to \mathcal{X}$, and its degeneration type on the boundaries of the formal fibre $\mathcal{X}$. In [Ka], K. Kato proved, using techniques from K-theory, a formula which compares the dimensions of the spaces of vanishing cycles in a finite morphism between formal germs of curves over a complete discrete valuation ring. To the best of my knowledge Kato's formula, which is analogous to the above formula in Theorem 3.4, is explicit only in the case where the morphism f is generically separable on the level of the special fibres. More precisely, the term d_s in Kato's formula is not explicitly computable in the case where we have a radical reduction type on the boundaries. The formula in Theorem 3.4 can be easily extended to the case of a Galois cover with group G which is nilpotent, or a group which has a normal p-Sylow subgroup. Our method to prove such a formula is to construct, using formal patching techniques à la Harbater, a compactification $\tilde{f} : Y \to X$ of the above cover $f : \mathcal{Y} \to \mathcal{X}$ (cf. Proposition 3.3.2). The formula follows then by comparing the genus of the special and the generic fibres of Y in this compactification.

As an application, the above formula gives interesting results when $\mathcal{X}$ is the formal germ of a semi-stable R-curve (cf. Sections 4.1 and 4.2); in particular, one can predict in this case whether $\mathcal{Y}$ is semi-stable or not. We give several examples which illustrate this situation, namely the case of Galois covers of degree p between formal germs of semi-stable R-curves (cf. Examples 4.1.3, 4.1.4, and 4.2.4). In particular, one can classify étale Galois covers of degree p between annuli (cf. Proposition 4.2.5). The above results and examples play a key role in [Sa-2] in order to exhibit and realise the "degeneration data" associated to Galois covers of degree p above a proper semi-stable R-curve.

1. Formal and rigid patching

In what follows, we explain the procedure which allows to construct (Galois) covers of curves in the setting of formal or rigid geometry by gluing together covers of formal affine or affinoid rigid curves with covers of formal fibres at closed points of the special fibre. We refer to the exposition in [Pr] for a discussion on patching results and for detailed references on the subject.

Let R be a complete discrete valuation ring with fractions field K, residue field k, and uniformiser π. Let X be an admissible formal R-scheme which is an *R-curve*, by which we mean that the special fibre $X_k := X \times_R k$ is a reduced one-dimensional k-scheme of finite type. Let Z be a finite set of closed points of X_k. For a point $x \in Z$, let $X_x := \operatorname{Spf} \widehat{O}_{X,x}$ be the formal completion of X at x, which is the *formal fibre* at the point x. Also let X' be a formal open subset of X whose special fibre is $X_k - Z$. For each point $x \in Z$, let $\{\wp_i\}_{i=1}^n$ be the set of minimal prime ideals of $\widehat{\mathcal{O}}_{X,x}$ which contain π, corresponding to the *branches* $(\eta_i)_{i=1}^n$ of the completion of X_k at x, and let $X_{x,i} := \operatorname{Spf} \widehat{\mathcal{O}}_{x,\wp_i}$ be the formal completion of the localisation of X_x at $\wp_i$. The ring $\widehat{\mathcal{O}}_{x,\wp_i}$ is a complete discrete valuation ring. The set $\{X_{x,i}\}_{i=1}^n$ is the set of *boundaries* of the formal fibre X_x. For each $i \in \{1,\dots,n\}$ we have a canonical morphism $X_{x,i} \to X_x$.

1.1. Definition. With the same notation as above a $(G\text{-})$*cover patching data* for the pair (X, Z) consists of the following:

(a) a finite (Galois) cover $Y' \to X'$ (with group G);
(b) for each point $x \in Z$, a finite (Galois) cover $Y_x \to X_x$ (with group G).

The above data (a) and (b) must satisfy to the following condition:

(c) if $\{X_{x,i}\}_{i=1}^n$ are the boundaries of the formal fibre at the point x, then for each $i \in \{1,\dots,n\}$ is given a (G-equivariant) X_x-isomorphism $\sigma_i : Y_x \times_{X_x} X_{x,i} \simeq Y' \times_{X'} X_{x,i}$.

1.2. Proposition (Formal patching). *Given a* $(G\text{-})$*cover patching data as in Definition* 1.1, *there exists a unique, up to isomorphism,* (*Galois*) *cover* $Y \to X$ (*with group* G) *which induces the above* (G-)*covers in* (a) (*respectively in* (b)) *when restricted to* X' (*respectively when pulled back to* X_x *for each point* $x \in Z$).

The proof of Proposition 1.2 is an easy consequence of Theorem 3.4 in [Pr], which is due to Ferrand and Raynaud.

1.3. With the same notation as above let $x \in Z$ and let $\widetilde{X}_k$ be the normalisation of X_k. There is a one-to-one correspondence between the set of points of $\widetilde{X}_k$ above x and the set of boundaries of the formal fibre at the point x. Let x_i be the point of $\widetilde{X}_k$ above x which corresponds to the boundary $X_{x,i}$, for $i \in \{1,\dots,n\}$. Assume that the point $x \in X_k(k)$ is rational. Then the completion of $\widetilde{X}_k$ at x_i is isomorphic to the spectrum of a ring of formal power series $k[[t_i]]$ in one variable over k, where t_i is a local parameter at x_i. The complete local ring $\widehat{\mathcal{O}}_{x,\wp_i}$ is a discrete valuation ring whose residue field is isomorphic to $k((t_i))$. Let T_i be an element of $\widehat{\mathcal{O}}_{x,\wp_i}$ which lifts t_i, such an element is called a *parameter* of $\widehat{\mathcal{O}}_{x,\wp_i}$. Then, it follows from [Bo] that there exists an isomorphism $\widehat{\mathcal{O}}_{x,\wp_i} \simeq R[[T_i]]\{T_i^{-1}\}$, where $R[[T_i]]\{T_i^{-1}\}$ is the ring of formal power series $\sum_{i\in\mathbb{Z}} a_i T^i$ with $\lim_{i\to-\infty} |a_i| = 0$, and where $|\ |$ is an absolute value of K associated to its valuation.

1.4. Rigid patching

The analogue of Proposition 1.2 is well known in rigid geometry, which is not surprising because of the relation between rigid and formal geometry (cf. [Ra,Bo-Lu]). We will briefly explain the patching procedure in this context locally. Let $\mathcal{X} := \operatorname{Spm} A$ be an affinoid reduced curve, and let X be a formal model of $\mathcal{X}$ with special fibre X_k. Let x be a closed point of X_k, and let $\mathcal{X}_x$ be the *formal fibre* of $\mathcal{X}$ at x, which is a non-quasi-compact rigid space and which consists of the set of points of $\mathcal{X}$ which specialise to the point x. The structure of the *boundary* of $\mathcal{X}_x$ is well known and depends canonically on the normalisation of the complete local ring $\widehat{\mathcal{O}}_{X_k,x}$ (cf. [Bo-Lu-1]). Namely this boundary decomposes into a disjoint union of *semi-open annuli* $\mathcal{X}_i := \mathcal{X}_{x,i}$ one corresponding to each minimal prime ideal η_i of $\widehat{\mathcal{O}}_{X_k,x}$. Consider the quasi-compact rigid space $\mathcal{X}' := \mathcal{X} - \mathcal{X}_x$. Let $f' : \mathcal{Y}' \to \mathcal{X}'$ and $f_x : \mathcal{Y}_x \to \mathcal{X}_x$ be (Galois)-covers (with group G). The (G-)cover $f' : \mathcal{Y}' \to \mathcal{X}'$ extends to (G-)covers $f_i : \mathcal{Y}_i \to \mathcal{X}_i$ of each of the components of the boundary of $\mathcal{X}_x$, and the germ of such an extension is unique (cf. [Ra]). A (G-)patching data in this context are (G-)equivariant isomorphisms between the germs of $f_i : \mathcal{Y}_i \to \mathcal{X}_i$ and the restriction of the initial (G-)cover $f_x : \mathcal{Y}_x \to \mathcal{X}_x$ to $\mathcal{X}_i$. The rigid patching result asserts that, given a (G-)patching data as above, then there exists a unique, up to isomorphism, (G-)cover $f : \mathcal{Y} \to \mathcal{X}$ which induces the above covers above $\mathcal{X}'$ and $\mathcal{X}_x$ when restricted to these analytic subspaces.

1.5. Local–global principle

As a direct consequence of the above patching results one obtains a *local–global principle*, for lifting of covers of curves, which is certainly well-known to the experts. More precisely, we have the following proposition.

1.6. Proposition. *Let X be a proper and flat algebraic* (*or formal*) *R-curve and let $Z := \{x_i\}_{i=1}^n$ be a finite set of closed points of X. Let $f_k : Y_k \to X_k$ be a finite generically separable* (*Galois*) *cover* (*of group G*) *whose branch locus is contained in Z. Assume that for each $i \in \{1, \dots, n\}$ there exists a* (*Galois*) *cover $f_i : Y_i \to \operatorname{Spf} \widehat{\mathcal{O}}_{X,x_i}$* (*of group G*) *which lifts the cover $\widehat{Y}_{k,i} \to \operatorname{Spec} \widehat{\mathcal{O}}_{X_k,x_i}$ induced by f_k, where $\widehat{Y}_{k,i}$ denotes the completion of Y_k above x_i. Then there exists a unique, up to isomorphism,* (*Galois*) *cover $f : Y \to X$* (*of group G*) *which lifts the cover f_k, and which is isomorphic to the cover f_i when pulled back to $\operatorname{Spf} \widehat{\mathcal{O}}_{X,x_i}$, for each $i \in \{1, \dots, n\}$.*

Proof. After passing to the formal completion of X along its special fibre we may reduce to the case where $\mathcal{X}$ is a formal R-curve. We treat the case where $Z = \{x\}$ consists of one point (the general case is similar). Let $U_k := \mathcal{X}_k - \{x\}$, and let $\mathcal{U}$ be a formal open in $\mathcal{X}$ having U_k as a special fibre. The étale cover $f'_k : V_k \to U_k$ induced by f_k above U_k can be uniquely lifted, by the lifting theorems of étale covers (cf. [Gr]), to an étale formal cover $f' : \mathcal{V} \to \mathcal{U}$. Let $\{\wp_i\}_{i=1}^n$ be the minimal prime ideals of $\widehat{\mathcal{O}}_{X,x}$ which contain π, and let $\mathcal{X}_i := \mathcal{X}_{x,i} := \operatorname{Spf} \widehat{\mathcal{O}}_{x,\wp_i}$ be the formal completion of the localisation of $\widehat{\mathcal{O}}_{X,x}$ at $\wp_i$. We have canonical morphisms $\mathcal{X}_i \to \mathcal{X}$, and $\mathcal{X}_i \to \operatorname{Spf} \widehat{\mathcal{O}}_{X,x}$. The cover f' (respectively the given cover $f_i : Y_i \to \operatorname{Spec} \widehat{\mathcal{O}}_{X,x_i}$) induces (by pull back) a cover $f'_i : \mathcal{Y}_i \to \mathcal{X}_i$ (respectively a cover $f_i : \mathcal{Y}_i \to \mathcal{X}_i$). By construction, for each $i \in \{1, \ldots, n\}$, the covers f_i and f_i' are isomorphic when restricted to the special fibre $\operatorname{Spec} k((t_i))$ of $\mathcal{X}_i$. Since both f_i and f_i' are étale and $\mathcal{X}_i$ is local and complete, we deduce that f_i and f_i' are isomorphic. Hence, we obtain a patching datum which allows us to patch the covers $f' : \mathcal{V} \to \mathcal{U}$ and $f_i : Y_i \to \operatorname{Spf} \widehat{\mathcal{O}}_{X,x_i}$ in order to obtain a cover $f : \mathcal{Y} \to \mathcal{X}$ with the required properties. Now, thanks to the formal GAGA theorem, this cover is algebraic $f : Y \to X$ and has the desired properties. Moreover, if the starting data are Galois then the constructed cover is also Galois with the same Galois group. $\square$

1.7. Remark. Although the formal patching result (Proposition 1.2) and the rigid patching result (Section 1.4) are equivalent, in this paper we opted for the use of the formal patching result and the framework of formal geometry since this seems to be more convenient for most readers. However, it should be clear that one could also use the framework of rigid geometry and adapt the content of this paper to this setting.

2. Degeneration of μ_p-torsors on the boundaries of formal fibres

In this section, we recall the results concerning the degeneration of μ_p-torsors from zero to positive characteristics as discussed in [Sa-1] and describe the degeneration type of such torsors above the boundaries of formal fibres of R-curves. We fix the following notation: R is a complete discrete valuation ring of unequal characteristic, with residue characteristic $p > 0$, and which contains a primitive pth root of unity ζ. We denote by K the fractions field of R, π a uniformising parameter, k its residue field, and $\lambda := \zeta - 1$. We also denote by v_K the valuation of K which is normalised by $v_K(\pi) = 1$.

2.1. Torsors under finite and flat R-group schemes of rank p: the group schemes $\mathcal{G}^n$ and $\mathcal{H}_n$

For any positive integer n consider the scheme $\mathcal{G}_R^n := \operatorname{Spec} R[x, 1/(\pi^n x + 1)]$: it is a commutative affine group scheme whose generic fibre is isomorphic to the multiplicative group $\mathbb{G}_m$ and whose special fibre is the additive group $\mathbb{G}_a$ (cf. [Oo-Se-Su] for more details). The group law of the group scheme $\mathcal{G}_R^n$ is the one for which the following map:

$$\alpha^{(n)} : \mathcal{G}_R^n \to \mathbb{G}_{m,R},$$

defined in terms of affine algebras by

$$R\big[u, u^{-1}\big] \to R\big[x, 1/(\pi^n x + 1)\big], \qquad u \to \pi^n x + 1,$$

is a group scheme homomorphism.

For $0 < n \leqslant v_K(\lambda)$, the polynomial $((\pi^n x + 1)^p - 1)/\pi^{pn}$ has coefficients in R and defines a group scheme homomorphism $\varphi_n : \mathcal{G}_R^n \to \mathcal{G}_R^{np}$ given by

$$R\big[x, 1/(\pi^{pn} x + 1)\big] \to R\big[x, 1/(\pi^n x + 1)\big], \qquad x \to \big((\pi^n x + 1)^p - 1\big)/\pi^{pn}.$$

The homomorphism φ_n is an isogeny of degree p, its kernel $\mathcal{H}_{n,R}$ is a finite and flat R-group scheme of rank p, its generic fibre is isomorphic to μ_p and its special fibre is either the k-radical group scheme α_p, if $0 < n < v_K(\lambda)$, or the étale k-group scheme $\mathbb{Z}/p\mathbb{Z}$, if $n = v_K(\lambda)$.

The following is a commutative diagram of exact sequences of sheaves in the fppf-topology:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{H}_{n,R} & \longrightarrow & \mathcal{G}_R^n & \xrightarrow{\varphi_n} & \mathcal{G}_R^{np} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \alpha^{(n)}} & & \downarrow{\scriptstyle \alpha^{(np)}} & & \\ 0 & \longrightarrow & \mu_{p,R} & \longrightarrow & \mathbb{G}_{m,R} & \xrightarrow{x^p} & \mathbb{G}_{m,R} & \longrightarrow & 0. \end{array}$$

The upper exact sequence in the above diagram has a Kummer exact sequence:

$$1 \to \mu_p \to \mathbb{G}_m \xrightarrow{x^p} \mathbb{G}_m \to 1 \tag{1}$$

as a generic fibre, and the exact sequence:

$$1 \to \alpha_p \to \mathbb{G}_a \xrightarrow{x^p} \mathbb{G}_a \to 1 \tag{2}$$

as a special fibre if $0 < n < v_K(\lambda)$, or an exact sequence of Artin–Schreier type:

$$0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{G}_a \xrightarrow{x^p - x} \mathbb{G}_a \to 0 \tag{3}$$

as a special fibre if $n = v_K(\lambda)$.

Let $\mathcal{U}$ be an R-scheme, and let $f : \mathcal{V} \to \mathcal{U}$ be a torsor under the group scheme $\mathcal{H}_{n,R}$. Then, there exists an open covering $(\mathcal{U}_i)$ of $\mathcal{U}$ and regular functions $u_i \in \Gamma(\mathcal{U}_i, \mathcal{O}_{\mathcal{U}})$, such that $\pi^{np} u_i + 1$ is defined up to multiplication by a p-power of the form $(1 + \pi v^n)^p$, and the torsor f is defined above $\mathcal{U}_i$ by an equation $T_i^p = (\pi^n T_i' + 1)^p = \pi^{np} u_i + 1$, where T_i and T_i' are indeterminates.

2.2. *Degeneration of μ_p-torsors on the boundaries of formal fibres*

In what follows, we explain the degeneration of μ_p-torsors on the boundary $\mathcal{X} \simeq \operatorname{Spf} R[[T]]\{T^{-1}\}$ of formal fibres of germs of formal R-curves. Here $R[[T]]\{T^{-1}\}$ denotes the ring of formal power series $\sum_{i \in \mathbb{Z}} a_i T^i$ with $\lim_{i \to -\infty} |a_i| = 0$, where $|\ |$ is an absolute value of K associated to its valuation. Note that $R[[T]]\{T^{-1}\}$ is a complete discrete valuation ring, with uniformising parameter π, and residue field $k((t))$ where $t \equiv T \bmod \pi$. The following result will be used in the next section in order to prove a formula comparing the dimensions of the spaces of vanishing cycles in a Galois cover of degree p between formal germs of R-curves.

2.3. Proposition. *Let $A := R[\![T]\!]\{T^{-1}\}$ (cf. the definition above), and let $f : \operatorname{Spf} B \to \operatorname{Spf} A$ be a non-trivial Galois cover of degree p. Assume that the ramification index of the corresponding extension of valuation rings equals 1. Then f is a torsor under a finite and flat R-group scheme G of rank p. Let δ be the degree of the different in the above extension. The following cases occur*:

(a) $\delta = v_K(p)$. *In this case, f is a torsor under $G = \mu_p$ and two cases can occur*:
  (a1) *For a suitable choice of the parameter T of A the torsor f is given, after eventually a finite extension of R, by an equation $Z^p = T^h$.*
  (a2) *For a suitable choice of the parameter T of A the torsor f is given, after eventually a finite extension of R, by an equation $Z^p = 1 + T^m$ where m is a positive integer prime to p.*

(b) $0 < \delta < v_K(p)$. *In this case, f is a torsor under the group scheme $G = \mathcal{H}_{n,R}$, where n is such that $\delta = v_K(p) - n(p-1)$. Moreover, for a suitable choice of the parameter T the torsor f is given, after eventually a finite extension of R, by an equation $Z^p = 1 + \pi^{np} T^m$, with $m \in \mathbb{Z}$ is prime to p.*

(c) $\delta = 0$. *In this case, f is an étale torsor under the R-group scheme $G = \mathcal{H}_{v_K(\lambda),R}$ and is given, after eventually a finite extension of R, by an equation $Z^p = \lambda^p T^m + 1$, where m is a negative integer prime to p, for a suitable choice of the parameter T of A.*

Proof. This proposition is a consequence of Lemma 2-16 in [Hy] where are classified all Galois extensions of degree p above a complete discrete valuation ring of unequal characteristics: $(0, p)$. Since we are assuming that the ramification index is 1, we are only concerned with the cases (b), (d), and (e) in [Hy] which correspond in the statement of our proposition to the cases (a), (b), and (c), respectively.

We start with the case (a). In this case, $\delta = v_K(p)$ and the torsor f is given by an equation $Z^p = u$ where $u = \sum_{i \in \mathbb{Z}} a_i T^i \in R[\![T]\!]\{T^{-1}\}$ is a unit (which is defined up to multiplication by a p-power) such that its image $\bar{u} := \sum_{i \geqslant l} \bar{a}_i t^i \in k((t))$ modulo π (here $a_i \equiv \bar{a}_i \bmod \pi$, and $l \in \mathbb{Z}$) is not a p-power. The torsor f is a torsor under the group scheme μ_p. Consider the *differential form* $\omega = \mathrm{d}\bar{u}/\bar{u}$. This is a well determined differential form on $\operatorname{Spec}(A/\pi A)$ which does not depend on the choice of u. Let $h := \operatorname{res}(\omega)$ be the residue of ω and $m := \operatorname{ord}(\omega) + 1$. We distinguish the following two cases:

(a1) $\bar{u} = \sum_{i \geqslant l} \bar{a}_i t^i \in k((t))$ with $l \neq 0$ is an integer prime to p, in which case l must be congruent to the residue of ω modulo p. Thus $\bar{u} = t^l \bar{v}$ where $\bar{v} \in k((t))$ with $\operatorname{ord}_t(\bar{v}) = 0$. In particular, $u = T^l v$ where $v \in R[\![T]\!]\{T^{-1}\}$ is a unit such that $v \equiv \bar{v} \bmod \pi$. After extracting an lth root of v, which requires in general an étale extension of R, and replacing the parameter T by $T v^{1/l}$, we may assume that the torsor f is given by an equation $u = T^l$. Finally, after multiplying by a suitable p-power, we obtain an equation of the form $u = T^h$, where $h \in \mathbb{F}_p^*$ is the residue class of l modulo p.

(a2) $\bar{u} = \sum_{i \geqslant l} \bar{a}_i t^i \in k((t))$ where l is a p power. After then multiplying u by a suitable p-power, we may assume that $\bar{u} = \bar{a}_0 + \bar{a}_1 t^p + \cdots + \bar{a}_{[m/p]} t^{[m/p]p} + \bar{a}_m t^m + \cdots$, where m prime to p equals $\operatorname{ord}(\omega) + 1$. After replacing successively u by $u(1 - a_i^{1/p} T^i)^p$ for $i \in \{1, \ldots, [m/p]\}$, which requires extracting pth roots of the a_i, we may assume that $a_i = 0$ for $i \in \{1, [m/p]\}$, in which case $\bar{u} = \bar{a}_0 + t^m \bar{v}$ where $\operatorname{ord}_t(\bar{v}) = 0$, and

$u = a_0 + T^m v$ where $v \equiv \bar{v} \bmod \pi$. After the extraction of a pth root of a_0, an mth root of v, replacing T by $Tv^{1/m}$ and multiplying u by a suitable p-power, we obtain finally an equation of the form $Z^p = 1 + T^m$.

(b) In this case, $0 < \delta = v_K(p) - n(p-1) < v_K(p)$, and f is a torsor under the group scheme $\mathcal{H}_{n,R}$ (cf. Section 2.1) defined by an equation $Z^p = 1 + \pi^{np} u$, where $u \in A$ is a unit such that its image $\bar{u}$ in $A/\pi A$ is not a p-power. The unit $1 + \pi^{np} u$ is defined up to multiplication by elements of the form $(1 + \pi^n v)^p$. Consider the *differential form* $\omega = \mathrm{d}\bar{u}$. This is a well defined differential form on $\operatorname{Spec}(A/\pi A)$ which does not depend on the choice of u. Let $h := \operatorname{res}(\omega)$ be the residue of ω and $m := \operatorname{ord}(\omega) + 1$. Let $\bar{u} = \sum_{i \geqslant l} \bar{a}_i t^i$. Then $m = \operatorname{ord}(\omega) + 1$ is the least integer i which is coprime to p. For $i < m$, and after extracting the pth roots of the a_i, we can replace $(1 + \pi^{np} u)$ by $(1 + \pi^{np} u)(1 + \pi^n(-a_1^{1/p} T^l - \cdots - a_{[m/p]}^{1/p} T^{[m/p]}))^p$ and thus reduce to the case where $l = m$ is coprime to p, so that $u = T^m v$, where $v \in A$ is such that $\operatorname{ord}_t(\bar{v}) = 0$. Finally, after extracting an m-root of v and replacing T by $Tv^{1/m}$, we obtain an equation of the form $Z^p = 1 + \pi^{np} T^m$.

(c) This case is treated as case (b). $\square$

2.4. Definition. With the same notation as in Proposition 2.3, we define the *reduction type* (or the degeneration type) of the torsor f to be $(G_k, -m, h)$, where $G_k := G \times_R k$ is the special fibre of the group scheme G, m is the "conductor" associated to the torsor $f_k : \operatorname{Spec} B/\pi B \to \operatorname{Spec} A/\pi A$, and $h \in \mathbb{F}_p$ its "residue," which are defined as follows: if G_k is radical as in cases (a) and (b), and if ω is the associated *differential form* (cf. proof of Proposition 2.3) then $h := \operatorname{res}(\omega)$ and $m := \operatorname{ord}(\omega) + 1$, and in case G_k is étale then $-m$ is the usual Hasse conductor associated to the corresponding étale extension of $k((t))$, in this case we set $h = 0$. In conclusion, following the notations of Proposition 2.3 the degeneration type is $(\mu_p, 0, h)$ in case (a1), $(\mu_p, -m, 0)$ in case (a2), $(\alpha_p, -m, 0)$ in case (b), and $((\mathbb{Z}/p\mathbb{Z}), -m, 0)$ in case (c).

2.5. Remark. The above corollary implies, in particular, that given two torsors above $A := R[\![T]\!]\{T^{-1}\}$, under a finite and flat R-group scheme of rank p, which have the same type of reduction (as defined in Definition 2.4) and the same degree of the different, then after eventually "adjusting" the Galois action on the Kummer generators of these two covers, and after eventually a finite extension of R, one can find a Galois-equivariant isomorphism between them. Also note that the above lemma can be easily adapted to the rigid setting in order to describe torsors under group schemes of rank p above the boundaries of formal fibres of curves in rigid geometry, which are semi-open annuli, as well as their degeneration type. The above corollary is also stated in [He] in a slight different form in terms of order p automorphisms of boundaries of formal fibres.

3. Computation of vanishing cycles

The main result of this section is Theorem 3.4, which gives a *formula* which compares the dimensions of the spaces of vanishing cycles in a Galois cover $\tilde{f} : \mathcal{Y} \to \mathcal{X}$ of group $\mathbb{Z}/p\mathbb{Z}$ between formal germs of R-curves, where R is a complete discrete valuation ring of inequal characteristics which contains a primitive pth root of unity, p being the residue characteristic, in terms of the degeneration type of $\tilde{f}$ above the boundaries of $\mathcal{X}$ as defined in Definition 2.4.

3.1. In this section, we consider a complete discrete valuation ring R of inequal characteristics, with residue characteristic $p > 0$, and which contains a primitive pth root of unity ζ. We denote by K the fraction field of R, π a uniformising parameter of R, k the residue field of R, and $\lambda := \zeta - 1$. We also denote by v_K the valuation of K which is normalised by $v_K(\pi) = 1$. *We assume that the residue field k is algebraically closed*. By a (formal) R-curve we mean a (formal) R-scheme of finite type which is normal, flat, and whose fibres have dimension 1. For an R-scheme X, we denote by $X_K := X \times_{\operatorname{Spec} R} \operatorname{Spec} K$ the *generic* fibre of X, and $X_k := X \times_{\operatorname{Spec} R} \operatorname{Spec} k$ its *special* fibre. In what follows, by a (*formal*) *germ* $\mathcal{X}$ of an R-curve we mean that $\mathcal{X} := \operatorname{Spec} \mathcal{O}_{X,x}$ is the (respectively $\mathcal{X} := \operatorname{Spf} \widehat{\mathcal{O}}_{X,x}$ is the formal completion of the) spectrum of the local ring of an R-curve X at a closed point x. Let $\mathcal{O}_x$ be the local ring of $\mathcal{X}_k$ at x. Let $\delta_x := \dim_k \widetilde{\mathcal{O}}_x/\mathcal{O}_x$ where $\widetilde{\mathcal{O}}_x$ is the normalisation of $\mathcal{O}_x$ in its total ring of fractions, and let r_x be the number of maximal ideals in $\widetilde{\mathcal{O}}_x$. The *contribution to the arithmetic genus* of the point x is by definition $g_x := \delta_x - r_x + 1$. We will call the integer g_x the *genus* of the point x. The following lemma is easy to prove (cf., for example, [Bo-Lu-1]).

3.1.1. Lemma. *Let X_k be a proper and reduced algebraic curve over k. Let $\widetilde{X}_k \to X_k$ be the normalisation of X_k, and let $\{X_i\}_{i\in I}$ be the irreducible components of $\widetilde{X}_k$. Let $\{x_j\}_{j\in J}$ be the set of singular points of X_k. Let $g(X_k)$ (respectively $g(X_i)$) be the arithmetic genus of X_k (respectively the arithmetic genus of X_i). Then $g(X_k) = \sum_{i\in I} g(X_i) + \sum_{j\in J} g_{x_j} + z(X_k)$, where $z(X_k)$ is the cyclomatic number of X_k as defined in* [Bo-Lu, *Definition* 4.2].

Note that $z(X_k) = 0$ if and only if the configuration of the irreducible components of X_k is a tree-like, and for each singular point x_j of X_k and every components X_i such that $x_j \in X_i$, the point x_j is unibranche on X_i. The case $z(X_k) = 0$ is the only one we will encounter in this paper.

3.2. Let $f : Y \to X$ be a finite cover between R-curves. Assume that the special fibres X_k and Y_k are reduced. Let y be a closed point of Y and let x be its image in X. Let $(x_j)_{j\in J}$ be the points of the normalisation $\widetilde{X}_k$ of X_k above x, and for a fixed j let $(y_{i,j})_{i\in I_j}$ be the points of the normalisation $\widetilde{Y}_k$ of Y_k which are above x_j. Assume that the morphism $f_k : Y_k \to X_k$ is *generically étale*. Under this assumption we have the following *local Riemann–Hurwitz formula*, which is due to Kato (cf. [Ka,Ma-Yo]):

$$(g_y + \delta_y - 1) = n(g_x + \delta_x - 1) + d_K - d_k^w, \tag{1}$$

where n is the local degree at y, which is the degree of the morphism $\operatorname{Spec} \widehat{\mathcal{O}}_{Y,y} \to \operatorname{Spec} \widehat{\mathcal{O}}_{X,x}$ between the completion of the local rings of Y (respectively X) at the point y (respectively x), d_K is the degree of the divisor of ramification in the morphism $\operatorname{Spec}(\widehat{\mathcal{O}}_{Y,y} \otimes_R K) \to \operatorname{Spec}(\widehat{\mathcal{O}}_{X,x} \otimes_R K)$. Let $d_{i,j}^w := v_{x_j}(\delta_{y_{i,j},x_j}) - e_{i,j} + 1$, where $\delta_{y_{i,j},x_j}$

is the discriminant ideal of the extension $\widehat{\mathcal{O}}_{\widetilde{X}_k,x_j} \to \widehat{\mathcal{O}}_{\widetilde{Y}_k,y_{i,j}}$ of complete discrete valuation ring, and $e_{i,j}$ its ramification index. The integer d_k^w is equal to the sum $\sum_{i,j} d_{i,j}^w$.

Actually, in [Ka] Kato proved, using techniques from K-theory, the existence of a formula analogous to (1) in general, without any assumption on the morphism $f_k : Y_k \to X_k$, where d_K is as above, $d_k^w = \sum_{i,j} d_{i,j}^w$ and $d_{i,j}^w$ is expressed as the "degree of the different" in a certain extension of two-dimensional discrete valuation rings. However, to the best of my knowledge, in the literature there is no explicit way to compute the term $d_{i,j}^w$. In Theorem 3.4 we propose an explicit formula similar to (1) in the case where f is Galois of group $\mathbb{Z}/p\mathbb{Z}$ and which includes the case where f_k is generically radical.

3.3. The compactification process

Let $\mathcal{X} := \operatorname{Spf} \widehat{\mathcal{O}}_{X,x}$ be the formal germ of an R-curve at a closed point x. Let $\tilde{f} : \mathcal{Y} \to \mathcal{X}$ be a Galois cover of group $\mathbb{Z}/p\mathbb{Z}$ with $\mathcal{Y}$ local. We assume that the special fibre of $\mathcal{Y}_k$ is reduced (this can always be achieved after a finite extension of R). We will construct a compactification of the above cover $\tilde{f}$, and as an application we will use this compactification in order to compute the arithmetic genus of the closed point of $\mathcal{Y}$. More precisely, we will construct a Galois cover $f : Y \to X$ of degree p, between proper algebraic R-curves, a closed point $y \in Y$, and its image $x = f(y)$, such that the formal germ of X (respectively equals Y) at x (respectively at y) equals $\mathcal{X}$ (respectively $\mathcal{Y}$), and such that the Galois cover $f_x : \operatorname{Spf} \widehat{\mathcal{O}}_{Y,y} \to \operatorname{Spf} \widehat{\mathcal{O}}_{X,x}$ induced by f between the formal germs at y and x is isomorphic to the above given cover $\tilde{f} : \mathcal{Y} \to \mathcal{X}$. The construction of such a compactification is well known in the case where $\tilde{f}$ has an étale reduction type on the boundaries (cf. [Ma-Yo,Ra-2]). In the case of radical reduction type of degree p on the boundaries, one is able to carry out such a construction using the formal patching result in Propositions 1.2 and 2.3. In fact, it suffices to be able to treat the case of one boundary, which is easily done using the next proposition and the formal patching result.

3.3.1. Proposition. *Let $D := \operatorname{Spf} R\langle 1/T\rangle$ be the formal closed disk centred at ∞ (cf.* [Bo-Lu, 1] *for the definition of $R\langle 1/T\rangle$). Let $\mathcal{D} := \operatorname{Spf} R[[T]]\{T^{-1}\}$, and let $\mathcal{D} \to D$ be the canonical morphism. Let $\tilde{f} : \mathcal{Y} \to \mathcal{D}$ be a non-trivial torsor under a finite and flat R-group scheme of rank p, such that the special fibre of $\mathcal{Y}$ is reduced. Then there exists a Galois cover $f : Y \to D$ with group $\mathbb{Z}/p\mathbb{Z}$ whose pull-back to $\mathcal{D}$ is isomorphic to the above given torsor $\tilde{f}$. More precisely, with the same notation introduced in Proposition* 2.3 *we have the following possibilities*:

(a1) *Consider the cover $f : Y \to D$ given generically by the equation $Z^p = T^{-r}$ where $r := p - h$. This cover is ramified at the generic fibre only above ∞. The finite morphism $f_k : Y_k \to X_k$ is a μ_p-torsor outside ∞, and Y_k is smooth above ∞. Moreover, the genus of the smooth compactification of Y_k equals* 0.

(a2) *Consider the cover $f : Y \to D$ given generically by the equation $Z^p = T^{-\alpha} \times (T^{-m} + 1)$, where α is an integer such that $m + \alpha \equiv 0 \bmod (p)$. This cover is ramified at the generic fibre only above ∞ and the distinct mth roots of unity. The finite morphism $f_k : Y_k \to D_k$ is a μ_p-torsor outside ∞ and the distinct mth root of* *unity, and Y_k is smooth. Moreover, the genus of the smooth compactification of Y_k equals* 0.

(b) *First if $m \leqslant 0$ consider the cover $f : Y \to D$ given generically by the equation $Z^p = 1 + \pi^{np} T^m$. This cover is a torsor under the group scheme $\mathcal{H}_{n,R}$, and its special fibre $f_k : Y_k \to X_k$ is a torsor under α_p. The affine curve Y_k has a unique singular point y which is the point above ∞ and $g_y = (-m-1)(p-1)/2$. Secondly, if $m \geqslant 0$, consider the cover $f : Y \to D$ given generically by an equation $Z^p = T^{-\alpha}(T^{-m} + \pi^{pn})$ where α is as in* (a2)*. This cover is ramified above ∞ and the m distinct roots of π^{-np}. The finite morphism $f_k : Y_k \to X_k$ is an α_p-torsor outside ∞, and the special fibre Y_k of Y is smooth. Moreover, in both cases the smooth compactification of the normalisation of Y_k has genus* 0.

(c) *Consider the cover $f : Y \to D$ given generically by the equation $Z^p = \lambda^p T^m + 1$. This cover is an étale torsor above D under the group scheme $\mathcal{H}_{v_K(\lambda)}$ and induces an étale torsor $f_k : Y_k \to D_k$ in reduction. Moreover, the genus of the smooth compactification of Y_k equals $(-m-1)(p-1)/2$.*

Proof. Case (c) is straightforward. In case (a1), one has only to justify that Y_k is smooth above ∞. The cover f is given by the equation $Z^p = T^{-r}$, and after using the Bezout identity one reduces to an equation $Z'^p = T^{-1}$ from which it follows that the complete local ring at the point above ∞ is $B = R[\![Z']\!]$ which shows that this point is smooth. In case (a2), one deduces in a similar way that Y_k is smooth. Case (b): if m is negative then the α_p-torsor $f_k : Y_k \to X_k$ is given, locally for the étale topology, above the point ∞, by an equation $z^p = t^{-m}$, where $t := T \bmod (\pi)$. The computation of the arithmetic genus of the singularity above infinity follows then by a direct calculation (cf., for example, [Ra-1], or [Sai, 2.9]). If m is positive, in order to see that Y_k is smooth, one considers the Galois cover $f' : \mathcal{Y}' \to \mathcal{P}$ above the formal R-projective line $\mathcal{P}$, obtained by gluing D with the formal unit closed disk $D' := \operatorname{Spf} R\langle T\rangle$ centred at 0, and given by the (same) equation $Z^p = 1 + \pi^{np} T^m$. The genus of the generic fibre of $\mathcal{Y}'$ is $(m-1)(p-1)/2$. The finite morphism $f'_k : \mathcal{Y}'_k \to \mathcal{P}_k$ is an α_p-torsor outside ∞, and outside 0 coincides with the morphism f_k. Above 0 this torsor is given, locally for the étale topology, by an equation $z^p = t^m$ where $t := T \bmod (\pi)$, hence the arithmetic genus above 0 equals $(m-1)(p-1)/2$ from which one deduces that Y_k is smooth. □

In the next proposition, we deal with the general case.

3.3.2. Proposition. *Let $\mathcal{X} := \operatorname{Spf} \widehat{\mathcal{O}}_x$ be the formal germ of an R-curve at a closed point x, and let $\{\mathcal{X}_i\}_{i=1}^n$ be the boundaries of $\mathcal{X}$. Let $\tilde{f} : \mathcal{Y} \to \mathcal{X}$ be a Galois cover with group $\mathbb{Z}/p\mathbb{Z}$ and with $\mathcal{Y}$ local. Assume that $\mathcal{Y}_k$ and $\mathcal{X}_k$ are reduced. Then there exists a Galois cover $f : Y \to X$ of degree p, between proper algebraic R-curves Y and X, a closed point $y \in Y$ and its image $x = f(y)$, such that the formal germ of X* (*respectively Y*) *at x* (*respectively at y*) *equals $\mathcal{X}$* (*respectively equals $\mathcal{Y}$*)*, and such that the Galois cover $\operatorname{Spf} \widehat{\mathcal{O}}_{Y,y} \to \operatorname{Spf} \widehat{\mathcal{O}}_{X,x}$ induced by f between the formal germs at y and x is isomorphic to the above given cover $\tilde{f} : \mathcal{Y} \to \mathcal{X}$. Moreover, the formal completion of X along its special fibre has a covering which consists of n closed formal disks D_i which are patched with*

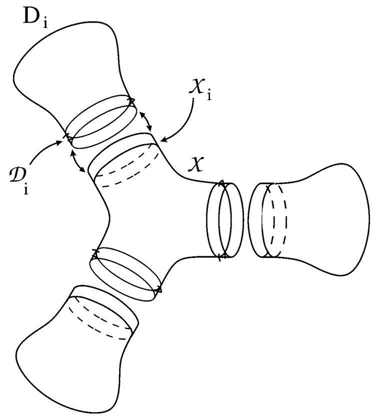

Fig. 1.

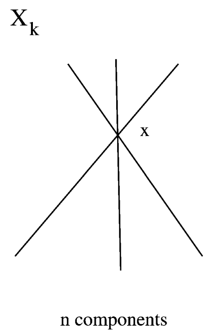

Fig. 2.

$\mathcal{X}$ along the boundaries $\mathcal{D}_i$, and the special fibre X_k of X consists of n smooth projective lines which intersect at the point x. In particular, the arithmetic genus of X_K equals g_x.

Proof. Let $\{\wp_i\}_{i=1}^n$ be the minimal prime ideals of $\widehat{\mathcal{O}}_x$ which contain π and which correspond to the branches $(\eta_i)_{i=1}^n$ of $\mathcal{X}_k$ at x, and let $\mathcal{D}_i := \operatorname{Spf} \widehat{\mathcal{O}}_{\wp_i}$ be the formal completion of the localisation of $\mathcal{X}$ at $\wp_i$. If T_i is a lifting of a uniformising parameter of the branch (η_i) of $\mathcal{X}_k$ at x then $\widehat{\mathcal{O}}_{\wp_i}$ is isomorphic to $R[[T_i]]\{T_i^{-1}\}$. For each $i \in \{1, \ldots, n\}$, consider a formal closed disk $D_i := \operatorname{Spf} R\langle 1/T_i \rangle$ centred at ∞, and the canonical morphism $\mathcal{D}_i := \operatorname{Spf} R[[T_i]]\{T_i^{-1}\} \to D_i$. As a consequence of the formal patching result, which is valid for coherent sheaves (cf. [Pr, Theorem 3.4]), one can patch $\mathcal{X} := \operatorname{Spf} \widehat{O}_x$ with the D_i, via the choice for each i of an automorphism of $\mathcal{D}_i$, in order to construct a proper formal R-curve X, and a closed point $x \in X$, such that the formal completion of X at x equals $\mathcal{X}$. The special fibre X_k of X consists by construction of a union of n smooth k-projective lines which intersect at the point x. Now the given cover $\tilde{f}$ induces a torsor $f_i : \mathcal{Y}_i \to \mathcal{D}_i$, under a finite and flat R-group scheme of rank p for each i, and by Proposition 3.3.2, one can find Galois covers $Y_i \to D_i$ of degree p which after pull back to

$\mathcal{D}_i$ coincide with f_i, for each $i \in \{1, \ldots, n\}$. The formal patching result (Proposition 1.2) again allows us then to patch these covers in order to construct a Galois cover $f : Y \to X$ of degree p with the desired properties. The formal R-curve is proper. By the formal GAGA theorems X is algebraic and the Galois cover $f : Y \to X$ is also algebraic. $\square$

The next result is the main one of this section, it provides an explicit *formula* which compares the dimensions of the spaces of vanishing cycles in a Galois cover of degree p between formal fibres.

3.4. Theorem. *Let $\mathcal{X} := \operatorname{Spf} \widehat{\mathcal{O}}_x$ be the formal germ of an R-curve at a closed point x, with $\mathcal{X}_k$ reduced. Let $\tilde{f} : \mathcal{Y} \to \mathcal{X}$ be a Galois cover with group $\mathbb{Z}/p\mathbb{Z}$, with $\mathcal{Y}$ local, and with $\mathcal{Y}_k$ reduced. Let $\{\wp_i\}_{i \in I}$ be the minimal prime ideals of $\widehat{\mathcal{O}}_x$ which contain π, and let $\mathcal{X}_i := \operatorname{Spf} \widehat{\mathcal{O}}_{\wp_i}$ be the formal completion of the localisation of $\mathcal{X}$ at $\wp_i$. For each $i \in I$, the above cover $\tilde{f}$ induces a torsor $\tilde{f}_i : \mathcal{Y}_i \to \mathcal{X}_i$, under a finite and flat R-group scheme of rank p, above the boundary $\mathcal{X}_i$ (cf. Proposition* 2.3*). Let $(G_{k,i}, m_i, h_i)$ be the reduction type of $\tilde{f}_i$ (cf. Definition* 2.4*). Let y be the closed point of $\mathcal{Y}$. Then one has the following* ***local Riemann–Hurwitz formula***:

$$2g_y - 2 = p(2g_x - 2) + d_\eta - d_s, \tag{2}$$

where d_η is the degree of the divisor of ramification in the morphism $\tilde{f}_K : \mathcal{Y}_K \to \mathcal{X}_K$ induced by $\tilde{f}$, here $\mathcal{X}_K := \operatorname{Spec}(\widehat{\mathcal{O}}_x \otimes_R K)$ and $\mathcal{Y}_K := \operatorname{Spec}(\widehat{\mathcal{O}}_{\mathcal{Y},y} \otimes_R K)$, and $d_s := \sum_{i \in I^{\mathrm{rad}}} (m_i - 1)(p-1) + \sum_{i \in I^{\mathrm{et}}} (m_i - 1)(p-1)$, where I^{rad} is the subset of I consisting of those i for which $G_{k,i}$ is radical, and I^{et} is the subset of I consisting of those i for which $G_{k,i}$ is étale and $m_i \neq 0$.

Proof. By Proposition 3.3.2, one can compactify the above morphism $\tilde{f}$. More precisely, in this proposition we constructed a Galois cover $f : Y \to X$ of degree p between proper algebraic R-curves, a closed point $y \in Y$, and its image $x = f(y)$, such that the formal germ of X (respectively of Y) at x (respectively at y) equals $\mathcal{X}$ (respectively equals $\mathcal{Y}$), and such that the Galois cover $\operatorname{Spf} \widehat{\mathcal{O}}_{Y,y} \to \operatorname{Spf} \widehat{\mathcal{O}}_{X,x}$, induced by f between the formal germs at y and x, is isomorphic to the given cover $\tilde{f} : \mathcal{Y} \to \mathcal{X}$. The special fibre of $\mathcal{X}$ consists (by construction) of card(I)-distinct smooth projective lines which intersect at the closed point x. The formal completion of X along its special fibre has a covering which consists of card(I) formal closed unit disks which are patched with the formal fibre $\mathcal{X}$ along the boundaries $\mathcal{X}_i$. The above formula (2) follows then by comparing the arithmetic genus of the generic fibre Y_K of Y and the arithmetic genus of the special fibre Y_k. Using the precise information given in Proposition 2.3.2, one can easily deduce that

$$g(Y_K) = pg_x + (1-p) + d_\eta/2 + \sum_{i \in I_a} (-m_i + 1)(p-1)/2 + \sum_{i \in I_{b,>}} (-m_i + 1)(p-1),$$

where I_a is the subset of I consisting of those i for which the degeneration data correspond to the one in case (a) of Proposition 2.3, and $I_{b,>}$ is the subset of I consisting of those i

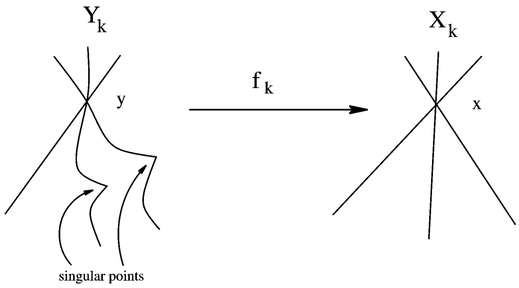

Fig. 3.

for which the degeneration data correspond to the one in case (b) of Proposition 2.3, with m positive. On the other hand, one has

$$g(Y_k) = g_y + \sum_{i \in I_{b,<}} (-m_i - 1)(p-1)/2 + \sum_{i \in I_{c,>}} (-m_i + 1)(p-1),$$

where $I_{b,<}$ is the subset of I consisting of those i for which the degeneration data correspond to the one in case (b) of Proposition 2.3 with $m_i < 0$, and $I_{c,<}$ is the subset of I consisting of those i for which the degeneration data correspond to the one in case (c) in Proposition 2.3, and with $m_i > 0$. Now, since Y is flat we obtain $g(Y_K) = g(Y_k)$ and the formula (2) directly follows. □

4. Examples of Galois covers of degree p above germs of curves

In what follows, we will keep the notations of Section 3.1. As a consequence of Theorem 3.4, we will deduce some results in this section in the case of a Galois cover $\mathcal{Y} \to \mathcal{X}$, where $\mathcal{X}$ is the formal germ of a semi-stable R-curve at a closed point. These results will play an important role in [Sa-2] in order to exhibit and realise the degeneration data which describe the semi-stable reduction of Galois covers of degree p.

4.1. We start with the case of a Galois cover of degree p above a germ of a *smooth* point.

4.1.1. Proposition. *Let $\mathcal{X} := \operatorname{Spf} R[[T]]$ be the germ of a formal R-curve at a smooth point x, and let $\mathcal{X}_\eta := \operatorname{Spf} R[[T]]\{T^{-1}\}$ be the boundary of $\mathcal{X}$. Let $f : \mathcal{Y} \to \mathcal{X}$ be a Galois cover of degree p, with $\mathcal{Y}$ local. Assume that the special fibre of $\mathcal{Y}$ is reduced. Let y be the unique closed point of $\mathcal{Y}_k$. Let $\delta_K := r(p-1)$ be the degree of the divisor of ramification in the morphism $f : \mathcal{Y}_K \to \mathcal{X}_K$. We distinguish two cases*:

(1) $\mathcal{Y}_k$ *is* unibranche *at y. Let (G_k, m, h) be the degeneration type of f above the boundary $\mathcal{X}_\eta$ (cf. Proposition 2.3). Then necessarily $r - m - 1 \geqslant 0$ and $g_y = (r - m - 1) \times (p-1)/2$.*

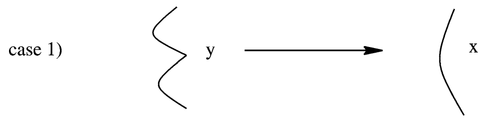

Fig. 4.

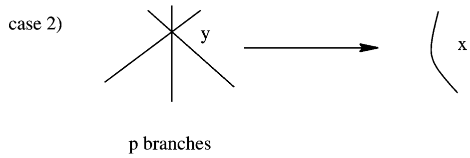

Fig. 5.

(2) $\mathcal{Y}_k$ *has* p-branches *at* y. *Then the cover* f *has an étale completely split reduction of type* $(\mathbb{Z}/p\mathbb{Z}, 0, 0)$ *on the boundary, i.e., the induced torsor above* $\operatorname{Spf} R[[T]]\{T^{-1}\}$ *is trivial, in which case* $g_y = (r-2)(p-1)/2$.

With the same notation as in Proposition 4.1.1, as an immediate consequence, one can immediately see whether the point y is smooth or not. More precisely, we have the following result.

4.1.2. Corollary. *We use the same notation as in Proposition* 4.1.1. *Then* y *is a smooth point, which is equivalent to* $g_y = 0$, *if and only if* $r = m + 1$. *This in the case of radical reduction type on the boundary is equivalent to* $r = -\operatorname{ord}(\omega)$, *where* ω *is the associated differential form* (*cf. Definition* 2.4). *In particular, if the reduction is of multiplicative type on the boundary, i.e., if* $G_k = \mu_p$, *then* $g_y = 0$ *only if* $r = 1$ *or* $r = 0$, *since in this case* $\operatorname{ord}(\omega) \geqslant -1$. *Also, if* $r = 1$ *and* $g_y = 0$ *then necessarily* $G_k = \mu_p$.

Next we will give examples of Galois covers of degree p above the formal germ of a smooth point which cover all the possibilities for the genus and the degeneration type on the boundary. Both in Examples 4.1.3 and 4.1.4 we use the same notations as in Proposition 4.1.1. We first begin with examples with genus 0.

4.1.3. Examples. The following are examples given by explicit equations of the different cases, depending on the possible degeneration type over the boundary, of Galois covers $f : \mathcal{Y} \to \mathcal{X}$ of degree p above $\mathcal{X} = \operatorname{Spf} R[[T]]$, and where $g_y = 0$ (here y denotes the closed point of $\mathcal{Y}$).

(1) For $m > 0$ an integer prime to p, consider the cover given generically by the equation $X^p = 1 + \lambda^p T^{-m}$. Here $r = m + 1$, and this cover has a reduction of type $(\mathbb{Z}/p\mathbb{Z}, m, 0)$ on the boundary.
(2) For $h \in \mathbb{F}_p^*$, consider the cover given generically by the equation $X^p = T^h$. Here $r = 1$, and this cover has a reduction of type $(\mu_p, 0, h)$ on the boundary.

(3) Consider the cover given generically by the equation $X^p = 1 + T$. Here $r = 0$, and this cover has a reduction of type $(\mu_p, -1, 0)$ on the boundary.
(4) For $n < v_K(\lambda)$, and $m < 0$, consider the cover given generically by the equation $X^p = 1 + \pi^{np} T^m$. Here $r = -m + 1$, and this cover has a reduction of type $(\alpha_p, -m, 0)$ on the boundary.
(5) For $n < v_K(\lambda)$, consider the cover given generically by the equation $X^p = 1 + \pi^{np} T$. Here $r = 0$, and this cover has a reduction of type $(\alpha_p, -1, 0)$ on the boundary.

Next we give examples of Galois covers of degree p above formal germs of smooth points which lead to a singularity with positive genus.

4.1.4. Examples. The following are examples given by explicit equations of the different cases, depending on the possible reduction type, of Galois covers $f : \mathcal{Y} \to \mathcal{X}$ of degree p above $\mathcal{X} = \operatorname{Spf} R[\![T]\!]$, and where $g_y > 0$.

(1) For $m > 0$ an integer prime to p, and $m' > m$, consider the cover given generically by the equation $X^p = 1 + \lambda^p (T^{-m} + \pi T^{-m'})$. Here $r = m' + 1$, and this cover has a reduction of type $(\mathbb{Z}/p\mathbb{Z}, m, 0)$ on the boundary. Moreover, the genus g_y of the closed point y of $\mathcal{Y}$ equals $(m' - m)(p - 1)/2$.
(2) For $h \in \mathbb{F}_p^*$ and $m > 0$ an integer prime to p, consider the cover given generically by the equation $X^p = T^{h'}(T^m + a)$, where h' is a positive integer such that $m + h' \equiv h \bmod (p)$ and $a \in \pi R$. Here $r = m + 1$, and this cover has a reduction of type $(\mu_p, 0, h)$ on the boundary. Moreover, the genus g_y of the closed point y of $\mathcal{Y}$ equals $m(p-1)/2$.
(3) For a positive integer m', an integer h such that $m' + h \equiv 0 \bmod (p)$, and $a \in \pi R$, consider the cover given generically by the equation $X^p = T^h (T^{m'} + a)(1 + T^m)$. Here $r = m' + 1$, and this cover has a reduction of type $(\mu_p, -m, 0)$ on the boundary. Moreover, the genus g_y of the closed point y of $\mathcal{Y}$ equals $(m' + m)(p - 1)/2$.
(4) For $n < v_K(\lambda)$ and integers $m > 0$ prime to p and $m < m'$, consider the cover given generically by the equation $X^p = 1 + \pi^{np}(T^{-m} + \pi T^{-m'})$. Here $r = m' + 1$, and this cover has a reduction of type $(\alpha_p, -m, 0)$ on the boundary. Moreover, the genus g_y of the closed point y of $\mathcal{Y}$ equals $(m' + m)(p - 1)/2$.

Note that in the cases (4) and (5) of Examples 4.1.3 and in the case (4) of Examples 4.1.4, in order to realise these covers above $\mathcal{X}$ for a given n, one needs, in general, to perform a ramified extension of R.

4.2. Next, we examine the case of Galois covers of degree p above formal germs at *double points*.

4.2.1. Proposition. *Let $\mathcal{X} := \operatorname{Spf} R[\![S, T]\!]/(ST - \pi^e)$ be the formal germ of an R-curve at an ordinary double point x of thickness e, and let $\mathcal{X}_1 := \operatorname{Spf} R[\![S]\!]\{S^{-1}\}$ and $\mathcal{X}_2 := \operatorname{Spf} R[\![T]\!]\{T^{-1}\}$ be the boundaries of $\mathcal{X}$. Let $f : \mathcal{Y} \to \mathcal{X}$ be a Galois cover with group $\mathbb{Z}/p\mathbb{Z}$ and with $\mathcal{Y}$ local. Assume that the special fibre of $\mathcal{Y}$ is reduced. We assume that $\mathcal{Y}_k$ has* two branches *at the point y. Let $\delta_K := r(p - 1)$ be the degree of the divisor of ramification in the morphism $f : Y_K \to X_K$. Let $(G_{k,i}, m_i, h_i)$ be the degeneration*

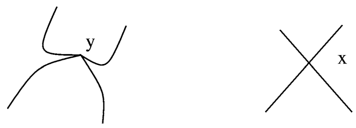

Fig. 6.

type on the boundaries of $\mathcal{X}$, *for* $i = 1, 2$. *Then necessarily* $r - m_1 - m_2 \geqslant 0$, *and* $g_y = (r - m_1 - m_2)(p - 1)/2$.

4.2.2. Proposition. *We use the same notation as in Proposition* 4.2.1. *We consider the remaining cases*:

(1) $\mathcal{Y}_k$ has $p + 1$ branches at y, *in which case we can assume that* $\mathcal{Y}$ *is completely split above* $\mathcal{X}_1$. *Let* $(G_{k,2}, m_2, h_2)$ *be the reduction type on the second boundary* $\mathcal{X}_2$ *of* $\mathcal{X}$. *Then necessarily* $r - m_2 - 1 \geqslant 0$ *and* $g_y = (r - m_2 - 1)(p - 1)/2$.
(2) $\mathcal{Y}_k$ has $2p$ branches at y, *in which case* $\mathcal{Y}$ *is completely split above the two boundaries of* $\mathcal{X}$ *and* $g_y = (r - 2)(p - 2)/2$.

With the same notation as in Proposition 4.2.1, and as an immediate consequence, one can recognise whether the point y is a double point or not. More precisely, we have the following corollary.

4.2.3. Corollary. *We use the same notations as in Proposition* 4.2.1. *Then* y *is an ordinary double point, which is equivalent to* $g_y = 0$, *if and only if* x *is an ordinary double point of thickness divisible by* p, *and* $r = m_1 + m_2$. *Moreover, if* $g_y = 0$, $r = 0$, *and if* $(G_{k,i}, m_i, h_i)$ *is the reduction type on the boundary for* $i = 1, 2$, *then necessarily* $h_1 + h_2 = 0$.

Proof. We need only to justify the last assertion. If both h_1 and h_2 equals 0 there is nothing to prove. Otherwise assume $h_1 \neq 0$. Then $m_1 = m_2 = 0$ (because $g_y = 0 = m_1 + m_2$), and one easily sees that necessarily $G_{k,1} = G_{k,2} = \mu_p$. So the cover $f : \mathcal{Y} \to \mathcal{X}$ is in this case a μ_p-torsor and its reduction $f_k : \mathcal{Y}_k \to \mathcal{X}_k$ is also a μ_p-torsor given by an equation $t^p = u$, and $\omega := \mathrm{d}u/u$ is the associated differential form. The restriction ω_i of ω to the ith branch, for $i = 1, 2$, is the differential form associated to the μ_p-torsor $f_i : \mathcal{Y}_i \to \mathcal{X}_i$ induced by f above the boundary $\mathcal{X}_i$ of $\mathcal{X}$. The equality $h_1 + h_2 = 0$ follows then from the fact that the sum of the residues of ω on each branch equals 0, which is a property of the regular differential forms at a double point. □

Next we give examples of Galois covers of degree p, above the formal germ of a double point, which lead to singularities with genus 0, i.e., double points, and such that $r = 0$. These examples will be used in [Sa-2] in order to realise the “degeneration data” corresponding to Galois covers of degree p.

4.2.4. Examples. The following are examples given by explicit equations of the different cases, depending on the possible type of reduction on the boundaries, of Galois covers $f : \mathcal{Y} \to \mathcal{X}$ of degree p above $\mathcal{X} = \operatorname{Spf} R[[S, T]]/(ST - \pi^e)$, with $r = 0$, and where $g_y = 0$, for a suitable choice of e and R. Note that $e = pt$ must be divisible by p. In all the following examples, we have $r = 0$.

(1) p-*Purity*: if f as above has an étale reduction type on the boundaries, and $r = 0$, then f is necessarily étale, and hence is completely split since $\mathcal{X}$ is strictly henselian.
(2) Consider the cover given generically by an equation $X^p = T^h$, which leads to a reduction on the boundaries of type $(\mu_p, 0, h)$ and $(\mu_p, 0, -h)$.
(3) For a fixed integer $m > 0$ prime to p, and after eventually a ramified extension of R, choose t such that $tm < v_K(\lambda)$ and consider the cover given generically by an equation $X^p = 1 + S^m$ which leads to a reduction on the boundaries of type $(\mu_p, -m, 0)$ and $(\alpha_p, m, 0)$.
(4) For a fixed integer $m > 0$ prime to p, and after eventually a ramified extension of R, choose t such that $t = v_K(\lambda)/m$ and consider the cover given generically by an equation $X^p = \lambda^p / T^m + 1$, which leads to a reduction on the boundaries of type $(\mathbb{Z}/p\mathbb{Z}, m, 0)$ and $(\mu_p, -m, 0)$.
(5) For a fixed integer $m > 0$ prime to p, and after eventually a ramified extension of R, choose t such that $tm < v_K(\lambda)$ and consider the cover given generically by an equation $X^p = 1 + \lambda^p S^{-m}$, which leads to a reduction on the boundaries of type $(\mathbb{Z}/p\mathbb{Z}, m, 0)$ and $(\alpha_p, -m, 0)$.
(6) For a fixed integer $m > 0$ prime to p, and after eventually a ramified extension of R, choose t and n such that $tm + n < v_K(\lambda)$ and consider the cover given generically by an equation $X^p = 1 + \pi^{np} S^m$, which leads to a reduction on the boundaries of type $(\alpha_p, -m, 0)$ and $(\alpha_p, m, 0)$.

In fact, one can describe Galois covers of degree p above formal germs of double points which are étale above the generic fibre and with genus 0. Namely, they are all of the form given in Examples 4.2.4. In particular, these covers are uniquely determined, up to isomorphism, by their degeneration type on the boundaries. More precisely, we have the following proposition.

4.2.5. Proposition. *Let $\mathcal{X}$ be the formal germ of an R-curve at an ordinary double point x. Let $f : \mathcal{Y} \to \mathcal{X}$ be a Galois cover of degree p, with $\mathcal{Y}_k$ reduced and local, and with $f_K : \mathcal{Y}_K \to \mathcal{X}_K$* étale. *Let $\mathcal{X}_i$ for $i = 1, 2$ be the boundaries of $\mathcal{X}$. Let $f_i : \mathcal{Y}_i \to \mathcal{X}_i$ be the torsors induced by f above $\mathcal{X}_i$, and let δ_i be the corresponding degree of the different* (*cf. Proposition* 2.3). *Let y be the closed point of $\mathcal{Y}$, and assume that $g_y = 0$. Then there exists an isomorphism $\mathcal{X} \simeq \operatorname{Spf} R[[S, T]]/(ST - \pi^{tp})$ such that, if say $\mathcal{X}_2$ is the boundary corresponding to the prime ideal (π, S), one of the following properties holds*:

(a) *The cover f is generically given by an equation $X^p = T^h$, with $h \in \mathbb{F}_p^*$. This cover leads to a reduction on the boundaries of $\mathcal{X}$ of type $(\mu_p, 0, h)$ and $(\mu_p, 0, -h)$. Here $t > 0$ can be any integer. In this case, $\delta_1 = \delta_2 = v_K(p)$.*
(b) *The cover f is generically given by an equation $X^p = 1 + T^m$ where $m > 0$ is an integer prime to p such that $tm < v_K(\lambda)$. This cover leads to a reduction on the boundaries of $\mathcal{X}$ of type $(\alpha_p, m, 0)$ and $(\mu_p, -m, 0)$. In this case, $\delta_2 = v_K(p) = \delta_1 + (p-1)tm$.*

(c) *The cover f is generically given by an equation $X^p = 1 + T^m$ where $m > 0$ is an integer prime to p such that $tm = v_K(\lambda)$. This cover leads to a reduction on the boundaries of $\mathcal{X}$ of type $(\mathbb{Z}/p\mathbb{Z}, m, 0)$ and $(\mu_p, -m, 0)$. In this case, $\delta_2 = v_K(p) = \delta_1 + (p-1)tm$ and $\delta_1 = 0$.*

(d) *The cover f is generically given by an equation $X^p = 1 + \lambda^p T^{-m}$ where $m > 0$ is an integer prime to p such that $tm < v_K(\lambda)$. This cover leads to a reduction on the boundaries of $\mathcal{X}$ of type $(\mathbb{Z}/p\mathbb{Z}, m, 0)$ and $(\alpha_p, -m, 0)$. In this case, $\delta_1 = \delta_2 + (p-1)tm$ and $\delta_2 = 0$.*

(e) *The cover f is generically given by an equation $X^p = 1 + \pi^{np} T^m$ for a positive integer m prime to p with $n + tm < v_K(\lambda)$, which leads to a reduction on the boundaries of type $(\alpha_p, -m, 0)$ and $(\alpha_p, m, 0)$. In this case, $\delta_2 = \delta_1 + (p-1)tm$ and $\delta_1 = (p-1)(v_K(\lambda) - (n + tm))$.*

Note that in all cases we have $\delta_2 - \delta_1 = mt(p-1)$.

Proof. We explain briefly the proof. Since $g_y = 0$, which is equivalent to the fact that y is a double point, the thickness $e = pt$ at the double point x of $\mathcal{X}$ is necessarily divisible by p. Set $\mathcal{X} \simeq \operatorname{Spf} A'$, where $A' = R[[T', S']]/(S'T' - \pi^{tp})$. The μ_p-torsor $f_K : \mathcal{Y}_K \to \mathcal{X}_K$ is given by an equation $X^p = u_K$ where u_K is a unit on $\mathcal{X}_K$. Such a unit can be uniquely written as $u_K := \pi^n T'^m u$, where n and m are integers, and $u \in R[[T', S']]/(S'T' - \pi^{tp})$ is a unit. Note first that we necessarily have $n \equiv 0 \bmod (p)$, since $\mathcal{Y}_k$ is reduced. Assume first that $\gcd(m, p) = 1$. Let $t' := T' \bmod (\pi)$, let $s' := S' \bmod (\pi)$, and let $\bar{u} := u \bmod (\pi)$ which is a unit of $k[[s', t']]/(s't')$. Let $T := T' u^{1/m}$. Then $\mathcal{X} \simeq \operatorname{Spf} A$, where $A = R[[S, T]]/(ST - \pi^{tp})$ for a suitable $S \in A'$. Let $B := A[X, Y]/(X^p - T^m, Y^p - S^m, XY - \pi^t)$. Then B is a finite and flat A-algebra which is integrally closed (because $B/\pi B$ is reduced), and $\mathcal{Y} = \operatorname{Spf} B$. The cover $f : \mathcal{Y} \to \mathcal{X}$ is thus generically given by an equation $X^p = T^m$ and, after multiplying T^m by a suitable p-power, we reduce to an equation of the form $X^p = T^h$ where $h \in \mathbb{F}_p^*$ and we are in case (a). Assume now that $m \equiv 0 \bmod (p)$. Here we have two cases:

(1) $\bar{u}$ is not a p-power. Then it is easy to see that after changing the Kummer generator of the torsor f_K, as in the proof of Proposition 2.3(a2), one can assume that u is such that $\bar{u} = 1 + t'^m \bar{v}$, where m is a positive integer prime to p and $\bar{v}$ is a unit of $k[[s', t']]/(s't')$. In particular, $\bar{u} = 1 + (t' \bar{v}^{1/m})^m = 1 + t^m$, where $t := t' \bar{v}^{1/m}$. Let $T := T' v^{1/m}$, where $v := (u - 1)/T'^m$. We have $\mathcal{X} \simeq \operatorname{Spf} R[[S, T]]/(ST - \pi^{tp})$ for a suitable $S \in A'$. The cover $f : \mathcal{Y} \to \mathcal{X}$ is thus generically given by an equation $X^p = 1 + T^m$, and we are in case (b). Let $\mathcal{X}_1 := \operatorname{Spf} R[[S]]\{S^{-1}\}$ be the boundary of $\mathcal{X}$ corresponding to the ideal (π, T). The torsor $f_1 : \mathcal{Y}_1 \to \mathcal{X}_1$ induced by f above X_1 is generically given by an equation $X^p = 1 + \pi^{pt} S^{-m}$, which implies that $t \leqslant v_K(\lambda)$, since f_1 is not completely split. Moreover, we are in case (b) if $t < v_K(\lambda)$, and in case (c) if $t = v_K(\lambda)$.

(2) $\bar{u}$ is a p-power. In this case, after changing the Kummer generator of the torsor f_K, we can assume that this torsor is given by an equation $X^p = 1 + \pi^{n'} v$, where $v \in A'$ does not belong to the ideal $\pi A'$, and v is not a p-power mod π. Also, after localisation and completion at the ideal (π, S') as above (namely, by studying the behaviour of the cover above the boundary $\mathcal{X}_1$), it follows from [Hy, Lemma 2-16] that necessarily $n' \leqslant p v_K(\lambda)$

and $n' = np$ is divisible by p. Moreover, after changing the Kummer generator of the torsor f_K as in the proof of Proposition 2.3(b), one can assume that $\bar{v} = t'^m \bar{v}'$, where m is an integer prime to p, and $\bar{v}'$ is a unit. Let $T := T'(v/T'^m)^{1/m}$. We have $\mathcal{X} \simeq \operatorname{Spf} R[\![S,T]\!]/(ST - \pi^{tp})$ for a suitable $S \in A'$. The cover $f : \mathcal{Y} \to \mathcal{X}$ is thus generically given by an equation $X^p = 1 + \pi^{np} T^m$, and we are in case (d) or (e). Let $\mathcal{X}_1$ be as above the boundary of $\mathcal{X}$ corresponding to the ideal (π, T). The torsor $f_1 : \mathcal{Y}_1 \to \mathcal{X}_1$ induced by f above X_1 is given by an equation $X^p = 1 + \pi^{p(n+t)} S^{-m}$, which imply that $t + n \leqslant v_K(\lambda)$ since f_1 is not completely split. Moreover, we are in case d if $t + n = v_K(\lambda)$ and in case e if $n + t < v_K(\lambda)$. $\square$

4.3. *Variation of the different*

(Compare with [He, 5.2].) The following result, which is a direct consequence of Proposition 4.2.5, describes the variation of the degree of the different from one boundary to another in a cover $f : \mathcal{Y} \to \mathcal{X}$.

4.3.1. Proposition. *Let $\mathcal{X}$ be the formal germ of an R-curve at an ordinary double point x. Let $f : \mathcal{Y} \to \mathcal{X}$ be a Galois cover of degree p, with $\mathcal{Y}_k$ reduced and local, and with $f_K : \mathcal{Y}_K \to \mathcal{X}_K$* étale. *Let y be the closed point of $\mathcal{Y}$. Assume that $g_y = 0$, which imply necessarily that the thickness $e = pt$ of the double point x is divisible by p. For each integer $0 < t' < t$, let $\mathcal{X}_{t'} \to \mathcal{X}$ be the blow-up of $\mathcal{X}$ at the ideal $(\pi^{pt'}, T)$. The special fibre of $\mathcal{X}_{t'}$ consists of a projective line $P_{t'}$ which meets two germs of double points x and x'. Let η be the generic point of $P_{t'}$, and let v_η be the corresponding discrete valuation of the function field of $\mathcal{X}$. Let $f_{t'} : \mathcal{Y}_{t'} \to \mathcal{X}_{t'}$ be the pull-back of f, which is a Galois cover of degree p, and let $\delta(t')$ be the degree of the different induced by this cover above v_η (cf. Proposition* 2.3*). Also, denote by $\mathcal{X}_i$, for $i = 1, 2$, the boundaries of $\mathcal{X}$. Let $f_i : \mathcal{Y}_i \to \mathcal{X}_i$ be the torsors induced by f above $\mathcal{X}_i$, let $(G_{k,i}, m_i, h_i)$ be their degeneration type, and let δ_i be the corresponding degree of the different (cf. Proposition* 2.3*). Say $\delta_1 = \delta(0)$, $\delta_2 = \delta(t)$, and $\delta(0) \leqslant \delta(t)$. We have $m := -m_1 = m_2$ say is positive. Then the following holds*:

(1) *We have $\delta(t') < v_K(p)$ for every $0 \leqslant t' \leqslant t$. In this case, for $0 \leqslant t_1 \leqslant t_2 \leqslant t$ we have $\delta(t_2) = \delta(t_1) + m(p-1)(t_2 - t_1)$ and $\delta(t')$ is an increasing function of t'.*
(2) *There exists $0 \leqslant t' \leqslant t$ such that $\delta(t') = v_K(p)$. In this case, there exists $0 \leqslant t_1 \leqslant t_2 \leqslant t$ such that $\delta(t') = v_K(p)$ is constant for $t_1 \leqslant t' \leqslant t_2$, $\delta(t')$ is increasing as t' increases from* 0 *to t_1, and $\delta(t')$ decreases as t' increases from t_2 to t.*

4.3.2. Remark. In [Gr-Ma] and [He] where studied order p-automorphisms of open p-adic disks and p-adic annuli. In their approach, one writes such an automorphism as a formal series and one deduce some results, e.g., the only if part of Corollary 4.1.2 and Corollary 4.2.3, using the Weierstrass preparation theorem. The approach adopted in this work, consisting of directly computing the vanishing cycles first, provides, I believe, another way to study such automorphisms. Namely, these are the covers above formal fibres of semi-stable R-curves of genus 0, and one can easily write down Kummer equations which lead to such covers as in Examples 4.1.3, 4.2.4, and Proposition 4.2.5.

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